Parametric Linear Fractional Programming for an Unbounded Feasible Region

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Abstract. Parametric analysis in linear fractional programming is significantly more complicated in case of an unbounded feasible region. We propose procedures which are based on a modified version of Martos' algorithm or a modification of Charnes–Cooper's algorithm, applying each to problems where either the objective function or the right-hand side is parametrized.

Key words. Fractional programming, parametric programming, unbounded feasible region.

1. Introduction

Linear fractional programs, though not convex, have the same local-global properties as linear (convex) programs. An extensive literature exists for linear fractional programs with fixed values for the parameters in the objective function and in the constraints [8], [36]. In applications however, parameters are sometimes subject to changes. Since the 1960's a good number of articles have appeared in parametric fractional programming [1]–[7], [9], [10], [14]–[20], [22], [24]–[28], [30]–[33], [35]–[45]. Most of these contributions assume a bounded feasible region. In this case the analysis is comparatively easy since the problem is equivalent to a parametric linear program which has thoroughly been researched.

In the unbounded case difficulties arise since a supremum may not be attained at a vertex but along an extreme ray. In this case a vertex following procedure cannot be used to solve the problem.

In [10] an algorithm is suggested for a linear fractional program with an unbounded feasible region where the right-hand side is parametrized. It is based on Dinkelbach's algorithm in fractional programming where numerator and denominator are separated by a parameter.

More recently, Cambini and Sodini [14], [38] proposed a procedure for fractional programs with a parameter in the numerator of the objective function which is based on a modified version of Martos' algorithm by Cambini and Martein [11].

In the present paper we follow this approach. We study the same parametric problem as in [38] by applying a modified Charnes–Cooper algorithm [21] instead. In the second part we present procedures for fractional programs parametrized on the right-hand side by using either a modified Martos algorithm or a modified Charnes–Cooper algorithm. As shown in Section 4, for a parametrized objective function the algorithm based on Charnes–Cooper's transformation is more suitable whereas for problems with a parameter on the right-hand side the use of the modified Martos algorithm is preferable.

2. The Linear Fractional Program

We consider the following linear fractional program:

$$\text{(P)} \quad \sup_{x \in X} f(x) = (c^T x + c_0)/(d^T x + d_0)\,, \tag{1}$$

where $X = \{x \in R^n \colon Ax = b, x \geq 0\}$, A is a $m \times n$ matrix, $c,, d \in R^n$, $b \in R^m$, c_0, $d_0 \in R$. We assume $X \neq \Phi$ and $d^T x + d_0 > 0 \quad \forall x \in X$.

For problem (P) the following three cases can occur:

(1) there exists an $x' \in X$ such that $\max\ \{f(x) \colon x \in X\} = f(x')$; in this case a vertex of X is an optimal solution;
(2) $\sup\{f(x) \colon x \in X\} = L < +\infty$, but the supremum is not attained; in this case there exists a vertex x' and an extreme ray r such that $\lim_{t \to +\infty} f(x' + tr) = L$;
(3) $\sup\{f(x) \colon x \in X\} = +\infty$; in this case there exists a vertex x' and an extreme ray r such that $\lim_{t \to +\infty} f(x' + tr) = +\infty$.

Clearly, if X is bounded, only case (1) occurs.

3. Algorithms

In this section we briefly describe two types of algorithms of fractional programming used in the parametric analysis below. These are the algorithm of Martos [29] and its modification by Cambini–Martein [11] and the algorithm by Charnes–Cooper [21].

Given a vertex x' of X and the corresponding base A_B, let

$$\begin{aligned} &c_N'^T = c_N^T - c_B^T A_B^{-1} A_N\,; \quad d_N'^T = d_N^T - d_B^T A_B^{-1} A_N\,, \\ &d_0' = d_B^T A_B^{-1} b + d_0, \quad c_0' = c_B^T A_B^{-1} b + c_0\,, \\ &\gamma = d_0' c_N' - c_0' d_N'\,. \end{aligned} \tag{2}$$

The algorithm of Martos

Step 0 Compute a basic feasible solution x'; *go to Step 1.*

Step 1 Compute c_N', d_N', c_0', d_0', γ. If $\gamma \leq 0$, *stop*; x' is optimal. Otherwise, select k such that $\gamma_k = \max\{\gamma_j\}$; *go to Step 2.*

Step 2 Compute $y^k = A_B^{-1} A_N^k$. If $y^k \leq 0$, then *stop*; $c_{N_k}'/d_{N_k}' = \sup\{f(x) \colon x \in X\}$. Otherwise, do a simplex iteration with x_{N_k} as entering variable. Let x' be the new basic solution; *go to Step 1.*

The algorithm of Martos guarantees the solution of the problem only if the feasible region is bounded [13], [38].

In order to overcome the limitations of Martos' algorithm in the unbounded case, Cambini and Martein have proposed a modification of it. The modified version of Martos' algorithm MVM differs from Martos' algorithm in the choice of the entering variable for simplex iterations. Only optimal level vertices are used. This modification is sufficient to eliminate the disadvantage of the algorithm of Martos.

In addition to the notation in (2), let

$$\gamma' = c'_N - f(x')d'_N . \tag{3}$$

The algorithm MVM of Cambini–Martein

Step 0 Solve problem $P_0: \min\{d^Tx + d_0 : x \in X\}$. If the optimal vertex x' of P_0 is unique, then *go to Step 1*. Otherwise, solve problem $P_1: \sup\{c^Tx + c_0 : x \in X'\}$ where $X' = X \cap \{x: d^Tx = d^Tx'\}$. If $\sup\{c^Tx + c_0 : x \in X'\} = +\infty$, then *stop*; $\sup\{f(x): x \in X\} = +\infty$. Otherwise *go to Step 1* with x' being an optimal vertex of P_1.

Step 1 Compute c'_N, d'_N, γ' and $J = \{j: \gamma'_j > 0\}$. If $J = \Phi$, then *stop*; x' is an optimal solution. Otherwise, select k such that $c'_{N_k}/d'_{N_k} = \max\{c'_{N_j}/d'_{N_j}: j \in J\}$; *go to Step 2*.

Step 2 Compute $y^k = A_B^{-1}A_N^k$. If $y^k \leq 0$, then *stop*; $c'_{N_k}/d'_{N_k} = \sup\{f(x): x \in X\}$. Otherwise, do a simplex iteration with x_{N_k} as entering variable. Let x' be the new basic solution; *go to Step 1*.

The algorithm of Charnes–Cooper

By using the transformation of variables $t = 1/(d^Tx + d_0)$, $y = tx$ Charnes–Cooper have shown that problem (P) is equivalent to the following one:

$$(\mathrm{P}') \qquad \begin{aligned} \sup \quad & c^Ty + c_0t \\ & Ay - bt = 0 \\ & d^Ty + d_0t = 1 \\ & y \geq 0 \\ & t > 0 \end{aligned} \tag{4}$$

If the feasible region X is a bounded set, we have $0 < p \leq d^Tx + d_0 < +\infty$, $\forall x \in X$, and $t = 0$, if permitted, could not occur in the optimal solution of problem (P′). Consequently, the constraint $t > 0$ is equivalent to the constraint $t \geq 0$, and thus problem (P′) turns out to be a linear program which can be solved by any algorithm for linear programs. On the other hand when the feasible region is unbounded, $t = 0$ can occur, and when it happens, (P) has a supremum obtained along an extreme ray [23], [34].

4. The Parametric Linear Fractional Program

Sensitivity and parametric analysis for a linear fractional problem with a bounded feasible region is similar to the one for linear programs [38]. However, when the feasible region is unbounded, some difficulties arise in studying the supremum of the problem (P) as a function of a parameter because of case (2) in Section 2 which cannot occur in linear programming.

4.1. PARAMETRIZATION OF THE OBJECTIVE FUNCTION

In this section we consider the parametrization of the numerator in the objective function of a linear fractional program. We point out that a problem with a parameter in the denominator can be reduced to one in the numerator by maximizing $-1/f(x)$ (assuming $f(x) > 0$). Let us consider the following parametric problem:

$$\mathrm{POBJ}(\theta) \quad z(\theta) = \sup_{x \in X} [((c + \theta u)^T x + c_0 + \theta u_0)/(d^T x + d_0)] \tag{5}$$

for some $(u, u_0) \in R^{n+1}$.

Now we describe a procedure for solving problem POBJ(θ) which is based on the algorithm MVM [38]. Following that, we will apply the Charnes–Cooper algorithm to the same problem. First, problem POBJ(0) is solved by means of algorithm MVM.

Let $x' = (A_B^{-1}b, 0)$ be the solution obtained by the algorithm. Clearly, x' is an optimal solution of the linear subproblem PL(0) where

$$\begin{aligned} \mathrm{PL}(\theta) \quad \sup \quad & (c + \theta u)^T x + c_0 + \theta u_0 \\ & x \in X\,, \quad d^T x = d^T x'\,. \end{aligned} \tag{6}$$

Let $c_{N'}^{*T} = c_{N'}^T - c_{B'}^T A_{B'}^{*-1} A_{N'}^* \leq 0$ be the vector of the reduced cost of x' for PL(0). Two cases can occur:

(1) $\gamma' \leq 0$; then x' is a maximum of POBJ(0);
(2) there exists an index k such that $\gamma'_k > 0$, $c'_{N_k}/d'_{Nk} = \max\{c'_{Nj}/d'_{Nj} : \gamma'_j > 0\}$ and $y^k = A_B^{-1}A_N^k \leq 0$; in this case POBJ(0) has a supremum equal to c'_{Nk}/d'_{Nk} on the extreme ray from x' along y^k.

Let *the stability set* of vertex x' denote the set of values of θ such that x' is a maximum or the supremum is attained on an extreme ray starting from x'. In order to find the stability set of x', c'_N, γ', c'_0, $c^*_{N'}$ are considered as a function of the parameter θ. Such functions are denoted by $c'_N(\theta)$, $\gamma'(\theta)$, $c'_0(\theta)$, $c'_{N'}(\theta)$. Let

$$H = \{\theta : \gamma'(\theta) \leq 0\}\,, \quad H' = \{\theta : c^*_{N'}(\theta) \leq 0\}\,. \tag{7}$$

Clearly, $H' \supseteq H$.

- If case (1) holds, $H \neq \Phi$ and for $\theta \in H$ x' is a maximum of POBJ(θ) with $z(\theta) = c'_0(\theta)/d'_0$.

– If case (2) holds, let us consider the set $H'' = \{\theta\colon c'_{Nk}(\theta)/d'_{Nk} = \max\{c'_{Nj}(\theta)/d'_{Nj}\colon \gamma'_j(\theta) > 0\}\}$; for $\theta \in H'(\theta) \cap H''(\theta)$ the supremum is attained along y^k with $z(\theta) = c'_{Nk}(\theta)/d'_{Nk}$.

Let $H' = \{\theta : \theta' \leq \theta \leq \theta''\}$. We will solve problem PL(θ^*) where $\theta^* = \theta' - \varepsilon$ ($\theta^* = \theta'' + \varepsilon$), $\varepsilon > 0$. Starting from x', by means of one simplex iteration the optimal solution x^* of POBJ(θ^*) is obtained. x^* lies on an edge of X and is not, in general, a vertex of X. We will find the best vertex (in terms of the value of the objective function) which belongs to the edge containing x^*. Let x'' be such a vertex. The stability set of vertex x'' is adjacent to the stability set of vertex x'.

Clearly, the described procedure allows us to find the function $z(\theta)$ through a vertex following examination. As regards to the properties of the paramatric function $z(\theta)$, it is easy to prove that $z(\theta)$ is a convex piecewise linear function [14]. The domain is the union of adjacent stability sets and hence convex. $z(\theta)$ is piecewise linear since a stability set is the union of sets of type H and H'' which are related to the same vertex of S. Thus $z(\theta)$ is linear for $\theta \in H$ ($\theta \in H''$) since the denominator is constant there. Convexity can be shown by verifying the defining inequality of convex functions. For details see [14].

As an alternative to the above approach, let us now apply the Charnes–Cooper transformation to the parametric problem POBJ(θ). Then the following parametric linear program is obtained:

$$\text{C–C}(\theta) \qquad \begin{aligned} f(\theta) &= \sup[(c + \theta u)^T y + (c_0 + \theta u_0)t] \\ Ay - bt &= 0\,, \\ d^T y + d_0 t &= 1\,, \quad y \geq 0\,, \quad t \geq 0\,. \end{aligned} \tag{8}$$

C–C(θ) is a standard parametric linear program and can be solved using the usual procedure for parametric linear programming. This method which is based on sensitivity analysis applied to reduced costs involves less operations than the modified Martos algorithm.

If in the optimal solution (y', t'), t' is greater than zero, then it is possible to transform the results into the space of variables x, namely $x' = y'/t'$. This is not possible when $t' = 0$. In this case the above transformation cannot be used. Let B' be the set of basic variables of the optimal solution of C–C(θ) and

$$A' = \begin{bmatrix} A & -b \\ d^T & d_0 \end{bmatrix}, \quad y_{B'} = A'^{-1}_{B'}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u_{B'} = A'^{-1}_{B'}\begin{bmatrix} -b \\ d_0 \end{bmatrix}.$$

Further, let $t_0 = y_{B'_k}/u_{B'_k} = \min\{y_{B'_i} : u_{B'_i} > 0\}$ and set $B = B' \backslash \{k\}$. It is easy to see [12] that the supremum of $z(\theta)$ is attained along the extreme ray starting from the vertex $x_B = (y_B - t_0 u_B)/t_0$ in the direction $d_B = -y_B/y_{B'_k}$. Clearly, these results allow us to determine the function $z(\theta)$ by solving problem $\mathrm{C} - \mathrm{C}(\theta)$.

EXAMPLE 1. Let us consider the following numerical example

$$z(\theta) = \sup\{[(-1+\theta)x_1 + (5-2\theta)x_2]/(x_1+2)\}$$
$$-x_1 + x_2 \leqslant 2\,, \quad x_1 - 2x_2 \leqslant 4\,,$$
$$x_1, x_2 \geqslant 0\,.$$

By using the Charnes–Cooper transformation we obtain the following parametric LP:

$$f(\theta) = \sup\{(-1+\theta)y_1 + (5-2\theta)y_2\}$$
$$-y_1 + y_2 - 2t + y_3 = 0\,,$$
$$y_1 - 2y_2 - 4t + y_4 = 0\,,$$
$$y_1 + 2t = 1\,,$$
$$y_1, y_2, y_3, y_4, t \geqslant 0\,.$$

The optimal solution for $\theta = 0$ is given by the following simplex Table I:

Table I

θ	2	y_1 1	y_2 0	y_3 2	y_4 0	t 0
$-z$	-5	-1	0	-5	0	0
t	1/2	1/2	0	0	0	1
y_2	1	0	1	1	0	0
y_4	4	3	0	2	1	0

Stability set: $-1+\theta \leqslant 0$, $-5+2\theta \leqslant 0$ imply $\theta \leqslant 1$; $z(\theta) = 5-2\theta$, $\theta \leqslant 1$ with the optimal solution $(y_1 = 0,\ y_2 = 1,\ t = 1/2)$. In the space of the variables x we have $(x_1 = 0,\ x_2 = 2)$.

For $\theta > 1$ the reduced cost of variable y_1 becomes positive and y_1 substitutes t in the basis. The new Table II is the following:

Table II

θ	1	y_1 0	y_2 0	y_3 2	y_4 0	t -2
$-z$	-4	0	0	-5	0	2
y_1	1	1	0	0	0	2
y_2	1	0	1	1	0	0
y_4	1	0	0	2	1	-6

Stability set: $2-2\theta \leqslant 0$, $-5+2\theta \leqslant 0$ imply $1 \leqslant \theta \leqslant 5/2$; $z(\theta) = 4-\theta$, $1 \leqslant \theta \leqslant 5/2$ with the optimal solution $(y_1 = 1,\ y_2 = 1,\ t = 0)$. In the space of the variables x we have $t_0 = \min\{1/2\} = 1/2$, $x_2 = 2$, $x_4 = (1+(1/2)6)/(1/2) = 8$, $d_B = (-1, -1)$.

For $\theta > 5/2$ the reduced cost of variable y_3 becomes positive and y_3 substitutes y_4 in the basis. The new Table III is the following:

Table III

θ	0	y_1 0	y_2 0	y_3 0	y_4 -1	t 4
$-z$	$-3/2$	0	0	0	$5/2$	-13
y_1	1	1	0	0	0	2
y_2	$1/2$	0	1	0	$-1/2$	3
y_3	$1/2$	0	0	1	$1/2$	-3

Stability set: $5/2-\theta\leq 0$, $-13+4\theta\leq 0$ imply $5/2\leq\theta\leq 13/4$; $z(\theta)=3/2$, $5/2\leq\theta\leq 13/4$ with the optimal solution $(y_1=1, y_2=1/2, t=0)$. In the space of the variables x we have $t_0=\min\{1/2, 1/6\}=1/6$, $x_1=(1-(1/6)2)/(1/6)=4$, $x_3=(1/2+(1/6)3)/(1/6)=6$, $d_B=(-2,-1)$.

For $\theta>13/4$ the reduced cost of variable t becomes positive and t substitutes y_2 in the basis. The new Table IV is the following:

Table IV

θ	$-2/3$	y_1 0	y_2 $-4/3$	y_3 0	y_4 $-1/3$	t 0
$-z$	$2/3$	0	$13/3$	0	$1/3$	0
y_1	$2/3$	1	$2/3$	0	$1/3$	0
t	$1/6$	0	$1/3$	0	$-1/6$	1
y_3	1	0	1	1	0	0

Stability set: $13/3-4/3\theta\leq 0$, $1/3-1/3\theta\leq 0$ imply $\theta\geq 13/4$; $z(\theta)=-2/3+2/3\theta$, $\theta\geq 13/4$ with the optimal solution $(y_1=2/3, y_3=1, t=1/6)$. In the space of the variables x we have $x_1=4$, $x_3=6$.

The function $z(\theta)$, depicted in Figure 1, is the following:

$$z(\theta)=\begin{cases}5-2\theta & \theta\leq 1\,,\\ 4-\theta & 1\leq\theta\leq 5/2\,,\\ 3/2 & 5/2\leq\theta\leq 13/4\,,\\ -2/3+2/3\theta & \theta\geq 13/4\,.\end{cases}$$

4.2. PARAMETRIZATION OF THE RIGHT-HAND SIDE

We now consider the following parametric linear fractional program:

$$\text{PRHS}\quad z(\theta)=\sup_{x\in X(\theta)}(c^Tx+c_0)/(d^Tx+d_0)\,, \tag{9}$$

where $X(\theta)=\{x\in R^n\colon Ax=b+\theta b^*, x\geq 0\}$ for some $b^*\in R^m$. We assume $d^Tx+d_0>0$ $\forall x\in X(\theta)$, $\forall\theta$.

This problem was solved in [10] by using the fractional programming algorithm of Dinkelbach in which numerator and denominator are separated by a parameter. Neither in Martos' nor in Charnes–Cooper's method such as parameter is needed. As in Section 4.1, we contrast the use of these latter two methods.

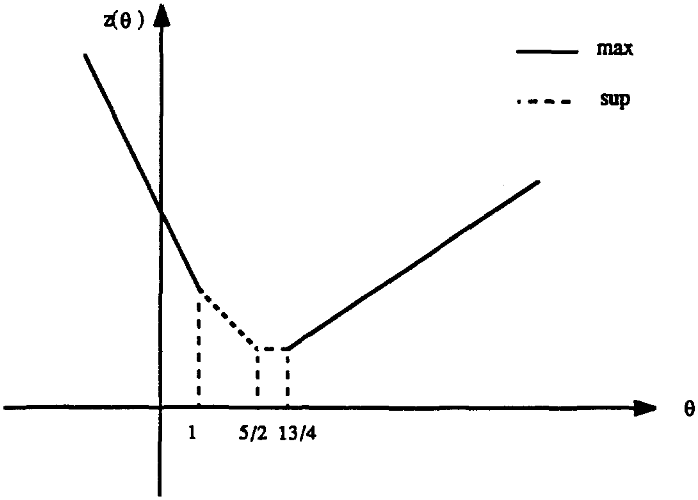

Fig. 1.

If we apply the Charnes–Cooper transformation to problem PRHS we obtain

$$\text{C}-\text{C}-\text{RHS}\qquad \begin{aligned} &f(\theta)=\sup[c^Ty+c_0t]\\ &Ay-(b+\theta b^*)t=0\,,\\ &d^Ty+d_0t=1\,,\quad y\geqslant 0\,,\quad t\geqslant 0\,. \end{aligned} \tag{10}$$

C–C–RHS is a parametric linear program with the parameter in the column of the matrix corresponding to the variable t. To solve such a problem is not easy. For this reason we investigate the application of the algorithm MVM to problem PRHS instead.

Given a basis A_B, let us define:

$$\begin{aligned} &c_N'^T=c_N^T-c_B^TA_B^{-1}A_N\,,\quad c_0'(\theta)=c_0+c_B^TA_B^{-1}(b+\theta b^*)\,,\\ &d_N'^T=d_N^T-d_B^TA_B^{-1}A_N\,,\quad d_0'(\theta)=d_0+d_B^TA_B^{-1}(b+\theta b^*)\,,\\ &x_B(\theta)=A_B^{-1}b+\theta A_B^{-1}b^*\,,\quad \gamma(\theta)=d_0'(\theta)c_N'-c_0'(\theta)d_N'\,. \end{aligned}$$

The vertex associated with the basis A_B is optimal if the following conditions hold:

$$\begin{aligned} &x_B(\theta)\geqslant 0\quad \text{(feasibility)}\,,\\ &\gamma(\theta)\leqslant 0\quad \text{(optimality)}\,. \end{aligned}$$

The supremum is attained on an extreme ray starting from the vertex $x_B(\theta)$ if the following conditions hold:

$$\begin{aligned} &x_B(\theta)\geqslant 0\,,\\ &\gamma_k(\theta)>0\,,\\ &A_B^{-1}A_N^k\leqslant 0\,,\\ &c'_{Nk}/d'_{Nk}=\max\{c'_{Nj}/d'_{Nj}:\ \ \gamma_j(\theta)>0\}\,. \end{aligned}$$

Suppose that problem PHRS is solved by means of algorithm MVM for a given value of θ and let A_B be the basis obtained.

If the vertex $x_B(\theta)$ is optimal, then define

$$H' = (\theta : x_B(\theta) \geq 0\} , \qquad H'' = \{\theta : \gamma(\theta) \leq 0\} ,$$
$$H^* = H' \cap H'' = \{\theta : \theta' \leq \theta \leq \theta''\} .$$

Clearly, for $\theta \in H^*$ the vertex $x_B(\theta)$ is optimal. For $\theta \not\subseteq H^*$ different cases can occur:

(1) feasibility is maintained while optimality is lost (i.e. $x_B(\theta) \geq 0$ while $\gamma(\theta) \nleq 0$);

(2) feasibility is lost while optimality is maintained (i.e. $x_B(\theta) \ngeq 0$ while $\gamma(\theta) \leq 0$);

(3) feasibility and optimality are lost (i.e. $x_B(\theta) \ngeq 0$ and $\gamma(\theta) \nleq 0$).

In case (1) there are two possibilities: (i) the supremum is attained along an extreme ray from $x_B(\theta)$ ($\gamma_k(\theta) > 0$, $A_B^{-1}A_N^k \leq 0$) and nothing must be done; (ii) the new optimal solution is an adjacent vertex of $x_B(\theta)$ which can be obtained by a primal simplex iteration. In case (2) the new optimal solution (if any) can be obtained by applying the dual simplex algorithm. In case (3) it is necessary to restart either feasibility or optimality and it is also necessary to maintain the optimality of level. This can be done by simplex iterations (dual or primal) until case (1) or (2) is obtained.

If the supremum is attained along an extreme ray from the vertex $x_B(\theta)$, then define

$$H' = \{\theta\colon x_B(\theta) \geq 0\} , \quad H'' = \{\theta\colon c'_{Nk}/d'_{Nk} = \max\{c'_{Nj}/d'_{Nj} : \gamma_j(\theta) > 0\} ,$$
$$H^* = H' \cap H'' = \{\theta\colon \theta' \leq \theta \leq \theta''\} .$$

Clearly, for $\theta \in H^*$ the extreme ray y^k from $x_B(\theta)$ is optimal. For $\theta \not\subseteq H^*$ different cases can occur:

(4) feasibility is maintained and optimality is obtained (i.e. $x_B(\theta) \geq 0$, $\gamma(\theta) \leq 0$);

(5) feasibility is maintained while optimality is lost (i.e. $x_B(\theta) \geq 0$ while $\gamma(\theta) \nleq 0$);

(6) feasibility and optimality are lost (i.e. $x_B(\theta) \ngeq 0$ and $\gamma(\theta) \nleq 0$).

In case (4) clearly $x_B(\theta)$ is an optimal vertex and nothing must be done. Case (5) is like case (1). Finally, case (6) is like case (3).

The proposed procedure is able to determine the function $z(\theta)$ for all values of θ.

EXAMPLE 2. Let us consider the following numerical example

$$z(\theta) = \sup[(-x_1 + 5x_2)/(x_1 + 2)]$$
$$-x_1 + x_2 \leq 2 + \theta , \quad x_1 - 2x_2 \leq 4 - 8\theta ,$$
$$x_1 , x_2 \geq 0 .$$

First, we solve the problem for $\theta = 0$. We obtain:

Table V

		x_1	x_2	x_3	x_4
N	$-10-5\theta$	4	0	-5	0
D	-2	1	0	0	0
x_2	$2+\theta$	-1	1	1	0
x_4	$8-6\theta$	-1	0	2	1

Then $\gamma(\theta)=(-2-5\theta, -10)\leqslant 0$, $H''=\{\theta:\theta\geqslant -2/5\}$, $x_B(\theta)=(2+\theta, 8-6\theta)\geqslant 0$, $H'=[-2, 4/3]$, hence $H^*=[-2/5, 4/3]$. Then for $\theta\in H^*$ $x^0=(0, 2+\theta)$ is optimal and $z(\theta)=(10+5\theta)/2$.

For $\theta>4/3$ we obtain Table VI:

Table VI

		x_1	x_2	x_3	x_4
N	$22-29\theta$	0	0	3	4
D	$6-6\theta$	0	0	2	1
x_2	$-6+7\theta$	0	1	-1	-1
x_1	$-8+6\theta$	1	0	-2	-1

Then $\gamma(\theta)=(26-40\theta,\ -2-5\theta)\leqslant 0$, $H''=\{\theta\colon \theta\geqslant 13/20\}$, $x_B(\theta)=(-6+7\theta, -8+6\theta)\geqslant 0$, $H'=\{\theta:\theta\geqslant 4/3\}$, hence $H^*=\{\theta:\theta\geqslant 4/3\}$. Then for $\theta\in H^* x^1=(-8+6\theta,\ -6+7\theta)$ is optimal and $z(\theta)=(29\theta-22)/(6\theta-6)$.

For $\theta<-2/5$ it follows from Table V that for $-2\leqslant\theta\leqslant -2/5$ the solution remains feasible while the reduced cost of variable x_1 is positive and the relative column is nonpositive. This implies that the supremum is reached along the following extreme ray:

$$\begin{bmatrix} x_2 \\ x_4 \end{bmatrix}=\begin{bmatrix} 2+\theta \\ 8-6\theta \end{bmatrix}+\begin{bmatrix} 1 \\ 1 \end{bmatrix}x_1 \quad \text{with} \quad z(\theta)=4\,.$$

For $\theta<-2$ with a pivot transformation in Table V we obtain Table VII.

Table VII

		x_1	x_2	x_3	x_4
N	$-2-\theta$	0	4	-1	0
D	θ	0	1	1	0
x_1	$-2-\theta$	1	-1	-1	0
x_4	$6-7\theta$	0	-1	1	1

Then $\gamma(\theta)=(-5\theta-2,\ -2), \gamma_2(\theta)>0$ for $\theta<-2$, $x_B(\theta)=(-2-\theta,\ 6-7\theta)\geqslant 0$, $H'=\{\theta:\theta\leqslant -2\}$, $H^*=H'$. Thus for $\theta\in H^*$ the supremum is reached along the following extreme ray:

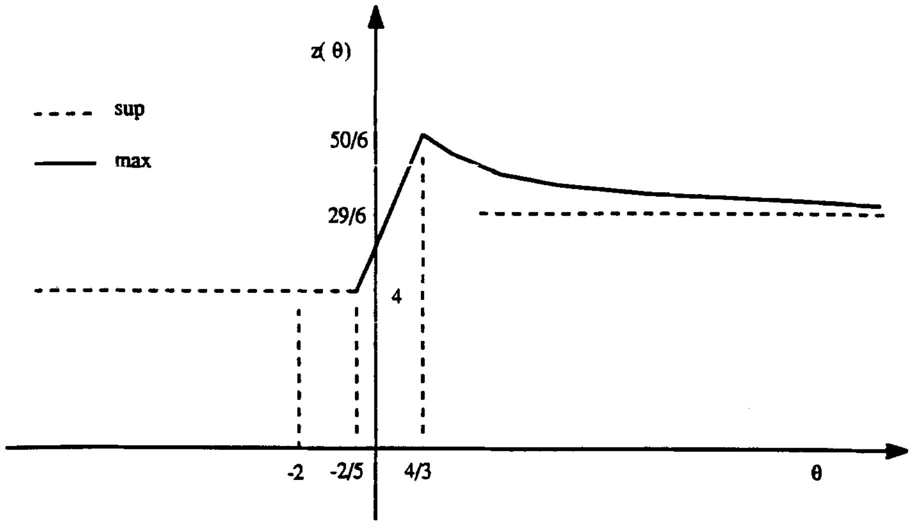

Fig. 2.

$$\begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2-\theta \\ 6-7\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2 \quad \text{with} \quad z(\theta) = 4 .$$

The function $z(\theta)$ is now completely described. In Figure 2 $z(\theta)$ is depicted. The function $z(\theta)$ is quasiconcave as it was shown for problem PRHS in general in [10].

5. Conclusion

Parametric linear fractional programming with an unbounded feasible region poses some difficulties not arising in the bounded case. Two types of algorithms have been applied to two kinds of parametric fractional programs. It turns out that a procedure based on the Charnes–Cooper transformation is more suitable in case of a parametrized objective function whereas a procedure based on the modified Martos algorithm is preferable in case of a parametrized right-hand side.

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References

1. Aggarwal, S. P. (1966), Stability of the Solution to a Linear Fractional Functional Programming Problem, *Zeitschrift für Angewandte Mathematik und Mechanik* **46**, 343–349.

2. Aggarwal, S. P. (1968), Parametric Linear Fractional Functional Programming, *Metrika* **12**, 106–114.
3. Aggarwal, S. P. (1970), Analysis of the Solution to a Linear Fractional Functional Programming Problem, *Metrika* **16**, 9–26.
4. Aggarwal, S. P. (1972), Variation in Parameters of Quadratic Fractional Functional Programming, *Revue Belge de Statistique d'Informatique et de Recherche Operationelle* **11**, 3–14.
5. Arora, S. R. and S. P. Aggarwal (1977), Linear Fractional Functional Programming with a Parameter in an Activity Vector, *Economic Computation and Economic Cybernetics and Research* **3**, 37–56.
6. Artjuhin, A.V. (1973), An Algorithm for the Solution of the Distribution Problem of Parametric Fractional Linear Programming (Russian), *Izdat*, "*Ilim*", *Frunze*, 30–36.
7. Artjuhin, A. V. (1973), Some Application of Parametric Fractional Linear Programming (Russian), *Izdat*, "*Ilim*", *Frunze*, 37–54.
8. Avriel, M., Diewert, W. E., Schaible, S., and I. Zang (1988), *Generalized Concavity*, Plenum Publishing Corporation, New York.
9. Bakhshi, H. C. (1979), Sensitivity Analysis in Linear Fractional Functional Programming Problem with Extreme Point Restriction, SCIMA, *Journal of Management Science and Applied Cybernetics* **8**, 6–15.
10. Bitran, G. and T. L. Magnanti (1976), Duality and Sensitivity Analysis for Fractional Programs, *Opns. Res.* **24**, 675–699.
11. Cambini, A. and L. Martein (1985), A Modified Version of Martos's Algorithm, *Methods of Oper. Res.* **53**, 33–44.
12. Cambini, A. and L. Martein (1992), Equivalences in Linear Fractional Programming, *Optimization* **23**, 41–51.
13. Cambini, A. and C. Sodini (1983), Sulla equivalenza di alcuni algoritmi di programmazione lineare frazionaria, Report A-105, Dip. di Mat., Univ. di Pisa.
14. Cambini, A. and C. Sodini (1987), On Parametric Linear Fractional Programs, Dip. di Stat. e Mat. Appl. all'Ec., Univ. di Pisa, Report n. 6.
15. Cernov, J. P. (1970), Several Problems of Parametric Fractional Linear Programming (Russian), *Optimal Planirovanie Vyp.* **16**, 98–111.
16. Cernov, J. P. (1970), A Certain Problem of Parametric Fractional Linear Programming (Russian), *Izvestja Akademii Nauk Kirgizskoi SSR* **3**, 20–27.
17. Chadha, S. S. (1971), A Simple Class of Parametric Linear Fractional Programs with Variable Coefficients, *Revue de la Faculté des Science de l'Université d'Istanbul*, Serie A36, 7–13.
18. Chadha, S. S. (1971), A Linear Fractional Functional Program with a Two Parameter Objective Function, *Zeitschrift für Angewandte Mathematik und Mechanik* **51**, 479–481.
19. Chadha, S. S. and S. Shivpuri (1973), A Simple Class of Parametric Linear Fractional Functional Programming, *Zeitschrift für Angewandte Mathematik und Mechanik* **53**, 644–646.
20. Chadha, S. S. and S. Shivpuri (1978), Multiparametric Linear Fractional Functional Programming, *Cahiers du Centre d'Etudes de Recherche Operationelle* **20**, 103–108.
21. Charnes, A. and W. W. Cooper (1962), Programming with Linear Fractional Functionals, *Nav. Res. Log. Quart.* **9**, 181–186.
22. Craven, B.D. (1981), Duality for Generalized Convex Fractional Programs, *Generalized Concavity in Optimization and Economics*, ed. by Schaible, S. and W. T. Ziemba, Academic Press, New York, 473–489.
23. Gasparotto, G. (1987), On the Charnes–Cooper Transformation in Linear Fractional Programming, Dip. di Stat. e Mat. Appl. all'Ec., Univ. di Pisa, Report n. 3.
24. Gupta, R. K. (1973), A Simple Class of Parametric Linear Fractional Functional Programming Problems, *Cahiers du Centre d'Etudes de Recherche Operationelle* **15**, 185–196.
25. Gupta, R. K. (1974), A Simple Class of Parametric Linear Fractional Functional Programming Problems, Erratum. *Cahiers du Centre d'Etudes de Recherche Operationelle* **16**, 179.
26. Kornbluth, J. S. H. and G. R. Salkin (1972), A Note on the Economic Interpretation of the Dual Variables in Linear Fractional Programming, *Zeitschrift für Angewandte Mathematik und Mechanik* **52**, 175–178.
27. Lange, E. G. and A. V. Artjuhin (1974), Some Problems of Linear Fractional Parametric Programming (Russian), Mathematical Methods for the Solution of Mathematical Economic Problems, *Izdat*, "*Ilim*", *Frunze*, 5–19.

28. Manjurov, D. M. (1968), Investigation of a Linear Fractional Programming Problem When Certain Parameters Are Varied (Russian), *Izd. Akad. Nauk Azerbaidzan SSR Ser. Fiz Tehn. Mat. Nauk*, 89–95.
29. Martos, B. (1964), Hyperbolic Programming, *Nav. Res. Log. Quart.* **11**, 135–155.
30. Pogodin, V. P. (1974), Study of a Certain Parametric Linear Fractional Programming Problem (Russian), Mathematical Methods in Economics Research, *Izdat, "Nauka", Moskow*, 44–50.
31. Prabha, S. (1982), Parametrizing a Column Vector in a Linear Fractional Programming Problem, *J. of Information and Optimization Sci.* **3**, 290–304.
32. Saxena, P. C. and S. P. Aggarwal (1980), Parametric Linear Fractional Functional Programming, *Economic Computation and Economic Cybernetics Studies and Research* **14**, 87–97.
33. Schaible, S. (1973), Fractional Programming: Transformations, Duality and Algorithmic Aspects, Technical Report 73–9, Operations Research House, Stanford University, November.
34. Schaible, S. (1974), Parameter-Free Convex Equivalent and Dual Programs of Fractional Programs, *Zeitschrift für Operations Research* **18**, 187–196.
35. Schaible, S. (1975), Marginalwerte in der Quotientenprogrammierung, *Zeitschrift für Betriebswirtschaft* **45**, 649–658.
36. Schaible, S. (1978), Analyse und Anwendungen von Quotientenprogrammen, ein Beitrag zur Planung mit Hilfe der nichtlinearen Programmierung, *Mathematical Systems in Economics* **42**, Hain-Verlag, Meisenheim.
37. Smyreva, N. V. (1974), An Algorithm for a Parametric Linear Fractional Programming Problem (Russian), *Optimizacija* **14**, 83–102.
38. Sodini, C. (1990), Equivalence and Parametric Analysis in Linear Fractional Programming, *Generalized Convexity and Fractional Programming with Economic Applications*, A. Cambini *et al.* (eds.), Proceedings, Pisa, Italy, 1988, Springer-Verlag, Heidelberg-New York, 143–154.
39. Swarup, K. (1968), On Varying All the Parameters in a Linear Fractional Functionals Programming Problem, *Metrika* **13**, 196–205.
40. Tammer, E. C. (1977), Parametrische hyperbolische Optimierungsprobleme mit Parametern in der Zielfunktion, *Mathematische Operationsforschung und Statistik, Series Optimization* **8**, 207–225.
41. *Varma, G. K.* (1969), *Some Aspects of Parametric Linear Fractional Programming, Journal of the Indian Statistical Association* **7**, 162–167.
42. Varma, G. K. (1972), General Parametric Linear Fractional Programming, *Metrika* **19**, 11–17.
43. Varma, G. K. (1972), On Parametric Linear Fractional Programming, *Trabajos Estadistica de Investigacion Operativa* **23**, 149–157.
44. Wadhwa, V. (1972), Linear Fractional Programs with Variable Coefficients, *Cahiers du Centre d'Etudes de Recherche Operationelle* **14**, 223–232.
45. Wadhwa, V. (1974), Parametric Linear Fractional Programming, SCIMA, *Journal of Management Science and Applied Cybernetics* **3**, 21–29.
46. Wagner, H. M. and J. S. C. Yuan (1968), Algorithm Equivalence in Linear Fractional Programming, *Management Science* **14**, 301–306.